A METHOD OF COMBINED SIMULATION OF UNSTEADY HEAT AND MASS TRANSFER PROCESSES

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A combined method is described for the electrical analog method of simulating heat and mass transfer processes, using two networks: On one network of capacitors and resistors, heat transfer is simulated; on the other, mass transfer, the cells of the networks being interconnected.

Methods of investigating and solving the systems of differential equations representing heat and mass transfer are based on the work of Lykov and his school. One such method is that of the electrical analog simulation of unsteady heat and mass transfer problems. It is known that the simulation of problems in heat transfer and mass transfer separately, without interconnection, is carried out on electrical integrators by creating the required boundary conditions at the boundary of an RC network. However, it is not possible to solve a heat and mass transfer problem by means of a single RC network.

In the present paper combined method is suggested for solving such problems on two networks: On one network of capacitors and resistors heat transfer is simulated; on the other, mass transfer, the cells of the networks being interconnected in such a way that a change in potential in one network influences the change in potential in the other. We shall call this method the method of combined simulation.

For definiteness we shall consider the equations of heat and mass transfer in capillary-porous substances, but a similar method can also be applied in other fields, for example, in simulating mass and heat transfer in solutions, binary gas mixtures, etc. Let the capillary-porous substance be full partly of liquid and partly of gas. Change of mass content at a certain point occurs as a result of phase transition and diffusion. Diffusion mass transfer arises from nonuniformity in concentration and from nonuniformity in temperature. During a change of heat content at each point, account is taken of flow rate of heat due to phase transition. The differential equations obeyed by the temperature $t(x, y, z, \tau)$ [°C] and by the mass transfer potential $\vartheta(x, y, z, \tau)$ [°M] have the form [1, 2]

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \frac{\varepsilon r C_m}{C_q} \frac{\partial \vartheta}{\partial \tau}, \tag{1}$$

$$\frac{\partial \vartheta}{\partial \tau} = a_m \nabla^2 \vartheta + a_m \delta_\Theta \nabla^2 t. \tag{2}$$

We shall examine two networks of resistors and capacitors C_1R_1 and C_2R_2 consisting of the same number of identically positioned cells (3). Each nodal point of the first network is coupled to the similarly located nodal point of the second network through a capacitor C. In the case of a one-dimensional problem, the electrical circuit has the form shown in Fig. 1a. The voltage v simulates the temperature $t = N_t v(I)$, and the voltage w simulates the mass transfer potential $\vartheta = N_\Theta w(II)$. The electrical parameters of a network may conventionally be regarded as being distributed along the axis $0x_e$. On the axis $0x_e$ a certain scale of units and a certain characteristic length of the analog, l_e, are chosen. The resistance of unit length of the first network is $R_1/\Delta x_e$, where Δx_e is the length of one cell of the circuit. $C_1/\Delta x_e$, $C_2/\Delta x_e$ are determined similarly. We shall consider, without introducing new notation, that R_1 is the resistance of unit length of the electrical circuit (or of unit cube in the case of a three-dimensional problem), and C_1 is the electrical capacitance, also of unit length (or of unit area in the case of a two-dimensional problem, etc.). We shall examine one cell of networks I and II in the case of a two-dimensional problem (Fig. 1b). As in the case of a one-dimensional problem, we shall choose axes $0x_e$ and $0y_e$. We shall examine a node in network I at which the voltage is v. On the basis of Kirchhoff's law, the sum of the input currents must be zero, i.e.,

$$i_1 - i_2 + i_3 - i_4 = i_5 - i_6. \tag{3}$$

The currents i_3, i_6 are proportional to the capacitances of the capacitors and to the derivative of potential difference across the plates of the capacitor:

$$i_5 = C_1 \Delta x_e \Delta y_e \frac{\partial v}{\partial \tau_e},$$

$$i_6 = C \Delta x_e \Delta y_e \frac{\partial (w - v)}{\partial \tau_e}.$$

Since, from Ohm's law, $i = -(\Delta y_e/R_1)(\partial v/\partial x_e)$ in the direction of the axis $0x_e$, and $i = -(\Delta x_e/R_1)(\partial v/\partial y_e)$ in the direction of the axis $0y_e$,

$$i_2 - i_1 = \frac{\partial i}{\partial x_e} \Delta x_e = -\frac{\Delta y_e}{R_1} \frac{\partial^2 v}{\partial x_e^2} \Delta x_e.$$

Similarly,

$$i_4 - i_3 = -\frac{\Delta x_e}{R_1} \frac{\partial^2 v}{\partial y_e^2} \Delta y_e.$$

Substituting the values of current found in (3), we have

$$\frac{\Delta y_e}{R_1}\frac{\partial^2 v}{\partial x_e^2}\Delta x_e + \frac{\Delta x_e}{R_1}\frac{\partial^2 v}{\partial y_e^2}\Delta y_e =$$

$$= C_1 \Delta x_e \Delta y_e \frac{\partial v}{\partial \tau_e} - C \Delta x_e \Delta y_e \frac{\partial (w - v)}{\partial \tau_e}.$$

In the second network, the equation for the potential of current is derived in a similar way, and we obtain the relation

$$\frac{\Delta y_e}{R_2}\frac{\partial^2 w}{\partial x_e^2}\Delta x_e + \frac{\Delta x_e}{R_2}\frac{\partial^2 w}{\partial y_e^2}\Delta y_e =$$

$$= C_2 \Delta x_e \Delta y_e \frac{\partial w}{\partial \tau_e} - C \Delta x_e \Delta y_e \frac{\partial (v - w)}{\partial \tau_e}.$$

These last relations give a system of equations relating to the potentials v and w:

$$\frac{\partial v}{\partial \tau_e} = \frac{1}{R_1 (C_1 + C)} \nabla^2 v + \frac{C}{C_1 + C}\frac{\partial w}{\partial \tau_e}, \quad (4)$$

$$\frac{\partial w}{\partial \tau_e} = \frac{1}{R_2 (C_2 + C)} \nabla^2 w + \frac{C}{C_2 + C}\frac{\partial v}{\partial \tau_e}. \quad (5)$$

The variables v and w, which simulate respectively temperature t and mass transfer potential ϑ, appear in Egs. (4) and (5) only in the form of partial derivatives, and therefore there may, generally speaking, be different zero potentials (i.e. "earth" potentials for networks I and II).

Equation (4), which simulates the equation of heat conduction (1), is indentical in structure to these last equations. We shall transform (5) by replacing $dv/d\tau_e$ by its value from (4). After arranging similar terms, we finally have

$$\frac{\partial v}{\partial \tau_e} = \frac{1}{R_1 (C_1 + C)} \nabla^2 v + \frac{C}{C_1 + C}\frac{\partial w}{\partial \tau_e}, \quad (6)$$

$$\frac{\partial w}{\partial \tau_e} = \frac{(C_1 + C)}{[(C_1 + C)(C_2 + C) - C^2] R_2} \nabla^2 w +$$

$$+ \frac{C}{[(C_1 + C)(C_2 + C) - C^2] R_1} \nabla^2 v. \quad (7)$$

This system of differential equations for the electrical potentials v and w is similar to the original heat and mass transfer equations. Equations (6) and (7) have been derived for the case of a two-dimensional model, but they are also valid for the one-dimensional scheme shown in Fig. 1a, and for a three-dimensional scheme; then the Laplace operator has the form

$$\nabla^2 v = \frac{\partial^2 v}{\partial x_e^2} + \frac{\partial^2 v}{\partial y_e^2} + \frac{\partial^2 v}{\partial z_e^2}.$$

We shall introduce similarity parameters of the heat and mass transfer equations (1) and (2):

$$\mathrm{Fo} = a_q \tau / l^2;\ \mathrm{Lu} = a_m / a_q;\ \mathrm{Ko}^* = \varepsilon r C_m \theta^* / C_q t^*;\ \mathrm{Pn} = \delta_\vartheta t^* / \vartheta^*.$$

In order to write (1) and (2) in dimensionless parameters, we shall introduce the relative magnitudes (simplexes)

$$X = x/l,\ Y = y/l;\ \Theta = \vartheta/\vartheta^*,\ T = t/t^*.$$

In terms of these dimensionless quantities, (1) and (2) take the form

$$\frac{\partial T}{\partial \mathrm{Fo}} = \nabla^2 T + \mathrm{Ko}^* \frac{\partial \Theta}{\partial \mathrm{Fo}}, \quad (8)$$

$$\frac{\partial \Theta}{\partial \mathrm{Fo}} = \mathrm{Lu}\, \nabla^2 \Theta + \mathrm{Lu}\, \mathrm{Pn}\, \nabla^2 T. \quad (9)$$

The Laplace operators in these equations correspond to partial derivatives also with respect to the dimensionless coordinates $X = x/l$, $Y = y/l$. The equations determine the two dependent variables ϑ and T from the independent ones Fo, X, Y (and Z in the case of a three-dimensional problem).

We shall convert Eqs. (6) and (7) also to dimensionless quantities in such a way that the equations obtained will be identical with (8) and (9). To do this, we shall transfer in the model to dimensionless coordinates and dimensionless potentials:

$$X = x_e/l_e,\ Y = y_e/l_e,\ V = v/v^*,\ W = w/w^*. \quad (10)$$

Then (6) takes the form

$$R_1 (C_1 + C)\, l_e^2 \frac{\partial V}{\partial \tau_e} = \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + \frac{C R_1 l_e^2 (C_1 + C) w^*}{(C_1 + C) v^*}\frac{\partial W}{\partial \tau_e}.$$

Introducing the two similarity parameters

$$\mathrm{Fo} = \frac{\tau_e}{R_1 (C_1 + C) l_e^2},\ \mathrm{Ko}^* = \frac{C}{C_1 + C}\frac{w^*}{v^*}, \quad (11)$$

the last equation takes the form

$$\frac{\partial V}{\partial \mathrm{Fo}} = \nabla^2 V + \mathrm{Ko}^* \frac{\partial W}{\partial \mathrm{Fo}}, \quad (12)$$

which is identical with (8).

If the assumed dimensionless variables and parameters (10) and (11) are inserted into the equation for the electrical potential w (7), it takes the form

$$\frac{\partial W}{\partial \mathrm{Fo}} = \frac{R_1 (C_1 + C)}{R_2 (C_2 + C)} \cdot \frac{1}{\left[1 - \dfrac{C^2}{(C_1 + C)(C_2 + C)}\right]} \nabla^2 W +$$

$$+ \frac{C}{(C_2 + C)\left[1 - \dfrac{C^2}{(C_1 + C)(C_2 + C)}\right]} \nabla^2 V. \quad (13)$$

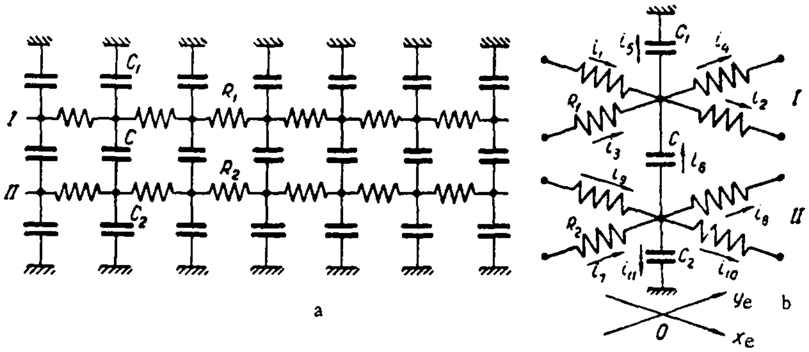

Fig. 1. Electrical diagram of the analog: a) for a one-dimensional problem; b) a cell of the networks for a two-dimensional problem.

In this relation the derivatives in the Laplace operator have been written, as they also were in (12), in terms of the dimensionless coordinates X and Y. To make (13) identical with (9), it is necessary that

$$\mathrm{Lu} = \frac{R_1(C_1+C)}{R_2(C_2+C)} \cdot \frac{1}{\left[1 - \frac{C^2}{(C_1+C)(C_2+C)}\right]},$$

and

$$\mathrm{Pn} = \frac{R_2 C v^*}{R_1 (C_1 + C) w^*},$$

which may be verified directly. The equation of electrical potential in the second network and in dimensionless coordinates takes the form

$$\frac{\partial W}{\partial \mathrm{Fo}} = \mathrm{Lu}\, \nabla^2 W + \mathrm{Lu}\, \mathrm{Pn}\, \nabla^2 V.$$

Thus, in order that the processes in the electrical analog and the heat and mass transfer processes shall be similar and be described by the same differential equations, it is necessary for the following four similarity parameters to be equal:

$$\mathrm{Fo} = a_q \tau / l^2 = \tau_e / R_1 (C_1 + C)\, l_e^2,$$

$$\mathrm{Lu} = \frac{a_m}{a_q} = \frac{R_1(C_1+C)}{R_2(C_2+C)\,[1 - C^2/(C_1+C)(C_2+C)]},$$

$$\mathrm{Ko}^* = \varepsilon r C_m \vartheta^* / C_q t^* = C w^* / (C_1 + C) v^*,$$

$$\mathrm{Pn} = \delta_\vartheta t^* / \vartheta^* = C R_2 v^* / (C_1 + C) R_1 w^*. \qquad (14)$$

The physical meaning of the similarity parameters Fo and Ko* becomes clear if we recall that

$$\tau = N_\tau \tau_e,\ l^2 = N_e^2 l_e^2,\ C_q = N_{C_q}(C_1 + C),\ \lambda_q = N_{\lambda_q} \frac{1}{R_1},$$

$$\vartheta = N_\vartheta w,\ t = N_t v.$$

At first sight it would seem that $\mathrm{Lu} = a_m/a_q$ should be equal to the ratio $R_1(C_1 + C)/R_2(C_2 + C)$ without the factor $[1-C^2/(C_1 + C)\cdot(C_2 + C)]$ in the denominator. But this is not so. In order to understand the mechanism of mass transfer in the analog, we shall examine the similarity of the heat and mass flux vectors j_q and j_m to the electric current vectors. In regard to the heat transfer process, this similarity is simple: The heat flux is

$$\mathbf{j}_q = -\lambda_q \nabla t. \qquad (15)$$

We shall introduce the dimensionless heat fluxes

$$\mathbf{I}_q = \mathbf{j}_q \cdot l / \lambda_q \cdot t^*.$$

Then (15) in dimensionless coordinates takes the form

$$\mathbf{I}_q = -\nabla T.$$

In the analog, in first network, the current i_1 satisfies the relation

$$\mathbf{i}_1 = -\frac{1}{R_1} \nabla v,$$

or, in dimensionless coordinates,

$$\mathbf{I} = \mathbf{i}_1 \cdot R_1 l_e / v^*,$$

and when the similarity parameters are equal, the equations are identical and the vectors must be equal, i.e.,

$$\mathbf{I}_q = \mathbf{j}_q \cdot l / \lambda_q \cdot t^* = \mathbf{i}_1 \cdot R_1 l_e / v^*.$$

On the other hand, the mass transfer vector is

$$\mathbf{j}_m = -\lambda_m \nabla \vartheta - \lambda_m \delta_\vartheta \nabla t. \qquad (16)$$

We shall introduce the dimensionless mass flux vector

$$\mathbf{I}_m = \mathbf{j}_m l / \lambda_m \vartheta^*.$$

Then (16) takes the form

$$\mathbf{I}_m = -\nabla \Theta - \mathrm{Pn} \nabla T. \qquad (17)$$

Taking into account that when similarity obtains

$$i_1 R_1 l/v^* = -\nabla V = -\nabla T,$$

and also,

$$i_2 R_2 l/w^* = -\nabla W = -\nabla \Theta,$$

which follows from the equality $i_2 = -\nabla w/R_2$, and we have, from (17)

$$I_m = i_2 R_2 l/w^* + \mathrm{Pn}\, i_1 R_1 l/v^*. \qquad (18)$$

Taking into account that in the analog $\mathrm{Pn} = CR_2 v^*/(C_1 + C)R_1 w^*$ (14), we have, from (18)

$$I_m = -\frac{l R_2}{w^*}\left(i_2 + \frac{C}{C_1 + C} i_1\right),$$

which means that the electric current simulating the mass flux passes not only through the second network, but also through the first, where it is reduced by the factor $C/(C_1 + C)$. Then the directions of i_2 and i_1 do not in general coincide.

NOTATION

τ) time; a_q) thermal diffusivity; ε) ratio of mass change due to a phase transition in the neighborhood of a certain point to the total mass change; r) specific heat of phase transition; C_m) isothermal mass capacity; C_q) heat capacity; a_m) potential conductivity of mass transfer; $\delta\Theta$) thermal gradient coefficient; l) characteristic length; t^* and ϑ^*) some specific temperature and mass transfer potential drops; v^* and w^*) some specific potential differences; Ko^*) modified Kossovich number.

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